This document is based on the article “From Population Dynamics to PDE”, to appear in The Mathematica Journal. There readers can have an interactive experience with the models for the evolution of populations presented below. The article provides a meaningful and intuitive introduction to the partial differential equations for traveling waves and diffusion to students in IQS. These PDEs are used to model signal transport within the cell during semester II of IQS. The students have worked with ODEs in the first semester of the course and a little bit with partial derivatives. It has only been necessary to supplement this reading with a part of one lecture during which students participate in a “stadium wave” and observe how their experience leads to the relationship

\[
\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x}.
\]

### Time Evolution of Profile Curves: An Introduction to Partial Differential Equations

- **Logistic Growth**

  Logistic growth of populations, with its characteristic sigmoidal solution curve, is modeled by the ordinary differential equation

  \[
  \frac{dy}{dt} = ky(A - y).
  \]

  You have seen this equation before in conjunction with the spread of infectious disease. In this context, \(y(t)\) represents the number of infected individuals at time \(t\). The parameter \(A\) represents the total population (the maximum possible number of infected persons), so that \(A - y\) is the number of uninfected persons. The parameter \(k\) controls the rate at which the disease spreads. More specifically we can think of \(k\) in terms of the probability (per unit time) that an infected person meets an uninfected person times the probability that the encounter leads to transfer of the disease from the infected to the uninfected. Thus \(k\) reflects the virulence of the particular disease.

  Later this semester you will see the same equation as a model for the kinetics of an autocatalytic reactions, where \(k\) will reflect the probability that two molecules come in contact to form product times the probability that the collision energy is greater than the activation energy necessary for the reaction to occur.

  In the language of ODE, the logistic model is set up as an initial value problem

  \[
  \frac{dy}{dt} = ky(A - y); \ k > 0, \ y_0 \geq 0.
  \]

  The objective is to find a function \(y(t)\) that satisfies the differential equation and has \(y(0) = y_0\). A graphical representation of the solution curve for certain values of \(k\), \(A\), and \(y_0\) is given below. At right is a phase diagram such as you have used to capture essential features of solutions to the logistic equation in terms of stable and unstable equilibria.
- **Time Evolution of a Profile Curve**

You have studied the size of the infected population, but the logistic model can be applied anytime the growth of a population is limited by the resources available to it. In this case, \(A\) is called the *carrying capacity*. Now consider the case of a population that is spread out along a narrow strip of land, which we will think of as a line. An initial study of the population might result in a function \(u_0(x)\) giving the size of the population at location \(x\) along the strip. But population size is a dynamic quantity so we should really employ a function of two variables, time and location, in our attempt to model its evolution. Thus the population size at time \(t\) and location \(x\) is \(u(t, x)\) with \(u(0, x) = u_0(x)\) recording the observed initial population distribution.

We model the time evolution of the population using the partial differential equation (PDE)

\[
\frac{\partial u}{\partial t} = k \ u \ (A(x) - u); \quad k, \ A(x) > 0, \quad u(0, x) = u_0(x) \geq 0.
\]

This is just a modification of the logistic equation in which the carrying capacity \(A\) now depends on location \(x\). This allows the new model to account for environmental conditions that vary from place to place. The function \(u = u(t, x)\) is now the unknown function that solves the PDE. In this setting, the role of the initial value, formerly played by the number \(u_0\), is being played by the initial function \(u(0, x) = u_0(x)\). Essentially what we have is a separate initial value problem at each location.

We will call the graph of \(u(0, x)\) the *initial profile curve*. Think of \(u(t, x)\) for a given value of \(t\) as the profile curve for the population at time \(t\). Its graph shows how the total population is distributed along the line at time \(t\). The left hand side of our PDE gives the time rate of change of the function \(u(t, x)\) and we will interpret the right hand side as the rule governing the time evolution of the initial profile curve \(u(0, x)\). This idea, the time evolution of a profile curve, is crucial to what follows. Together the next three graphs illustrate this point of view.
As you might expect, the initial population will evolve toward the carrying capacity. Here is the snapshot showing how the initial profile curve has evolved on its way toward the carrying capacity.

**Traveling Waves**

In the previous example, the time evolution of the profile curve was governed by a right hand side that involved only the values of the function $u(t, x)$. The next PDE we consider has the time evolution at location $x$ governed by the slope of the profile curve at $x$. The simplest equation of this type has the time rate of change of $u$ proportional to its slope,

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x},$$

with $u(0, x) = u_0(x)$, the initial profile curve, given as before. In class you saw this PDE in conjunction with the stadium wave, where $c$ controls the speed of the wave. Here we will think of the equation as a model for migration of the population along the strip of land. Once again the initial profile curve evolves in time, this time via a translation to the right at speed $c$. 
We showed in class how an application of the chain rule shows that \( u_0(x - ct) \) satisfies the migration equation for any initial population distribution \( u_0 \). In terms of cell signaling, the PDE \( \frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} \) is a simplified version of the transport equation that models how dissolved solids travel along with a current. Some chemical reactions exhibit this behavior even without a current. A YouTube video of the Belousov Zhabotinsky reaction is available at http://www.youtube.com/watch?v=DSaogqQLG_c & feature = related

### Dispersion

In response to overcrowding in one location, a population may disperse over a wider expanse. A model is provided by making the time evolution of the population graph proportional to the concavity of the graph. To get an idea of why this works, recall that a local maximum is characterized via the second derivative test as occurring where the tangent line is horizontal and the graph is concave down (negative second derivative). Thus the model stipulates that near a local maximum the future population will decrease. Similarly, the population will increase near a local minimum. We write the dispersion equation as

\[
\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2}; \quad d > 0,
\]

and refer to the proportionality constant \( d \) as the dispersion coefficient. The larger the dispersion coefficient, the faster the dispersion takes place.
In these graphs, the total population is represented by the area under the curve and does not change. As time moves forward, the dispersion process leads to a population that is, for practical purposes, uniformly distributed. In chemistry, our dispersion equation is called the diffusion equation. When signal is released, chemical concentrations naturally diffuse toward a uniform concentration throughout the cell. This will be our most important model for signal transport and we will link the model to the physical process of Brownian motion later in the semester.

**Summary**

In this reading you have seen how functions that depend on time and location are important for describing real phenomena of scientific interest. Mathematical models for such phenomena are expressed in terms of partial differential equations. The best way to visualize a PDE model is via the time evolution of an initial spatial profile curve. The three examples in the reading involve the three simplest ways to govern the time evolution:

\[
\frac{\partial u}{\partial t} = k \left( A(x) - u \right) \quad \text{has the evolution of } u(t, x) \text{ governed by the value of } u;
\]

\[
\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} \quad \text{has the evolution of } u(t, x) \text{ governed by the slope of } u;
\]

\[
\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} \quad \text{has the evolution of } u(t, x) \text{ governed by the concavity of } u.
\]

The simplest way to remember what has been presented here is to link the type of equation to its effect on the graph of the initial profile curve. In the context of cell signaling, the models link physical processes to chemical reactions to changes in biological
function. Our best example will occur later in the semester: a combined model applied to transmission of nerve impulses along an axon in which ions are transported across the membrane of the axon at the same time as they diffuse along the length of the axon.